Exercise 7

Show that the eigenvalues of the eigenvalue problem

$$\begin{aligned} u_{tt} + c^2 u_{xxxx} &= 0, \quad 0 < x < \ell,\ t > 0, \\ u(0,t) = 0 &= u(\ell,t) \quad \text{for } t \geq 0, \\ u_x(0,t) = 0 &= u_x(\ell,t) \quad \text{for } t \geq 0, \end{aligned}$$

satisfy the equation

$$\cos(\lambda\ell)\cosh(\lambda\ell) = 1.$$

Solution

The PDE and the boundary conditions are linear and homogeneous, so the method of separation of variables may be applied. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and substitute it into the PDE and boundary conditions:

$$X(x)T''(t) + c^2 X''''(x)T(t) = 0 \quad \rightarrow \quad -\frac{T''(t)}{c^2 T(t)} = \frac{X''''(x)}{X(x)} = k. \tag{1}$$

$$\begin{aligned} u(0,t) = 0 \quad &\rightarrow \quad X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0 \\ u(\ell,t) = 0 \quad &\rightarrow \quad X(\ell)T(t) = 0 \quad \rightarrow \quad X(\ell) = 0 \\ u_x(0,t) = 0 \quad &\rightarrow \quad X'(0)T(t) = 0 \quad \rightarrow \quad X'(0) = 0 \\ u_x(\ell,t) = 0 \quad &\rightarrow \quad X'(\ell)T(t) = 0 \quad \rightarrow \quad X'(\ell) = 0 \end{aligned}$$

The left side of (1) is a function of t, and the right side is a function of x. Therefore, both sides must be equal to a constant. This constant must be positive so that the solution to $T''(t) = -kc^2 T(t)$ remains finite as $t \to \infty$. Let $k = \lambda^4$; the reason for choosing λ^4 instead of λ^2 is to make the equation for $X(x)$ more convenient to solve.

$$\frac{d^4 X}{dx^4} - \lambda^4 X = 0, \quad X(0) = 0,\ X(\ell) = 0,\ X'(0) = 0,\ X'(\ell) = 0$$

This is a linear homogeneous ordinary differential equation with constant coefficients, so the solution has the form, $X(x) = e^{rx}$. Substituting this into the equation gives

$$\begin{aligned} &r^4 e^{rx} - \lambda^4 e^{rx} = 0 \\ &e^{rx}(r^4 - \lambda^4) = 0 \\ &r^4 - \lambda^4 = 0 \\ &(r^2 + \lambda^2)(r^2 - \lambda^2) = 0 \\ &(r + i\lambda)(r - i\lambda)(r + \lambda)(r - \lambda) = 0 \\ &\rightarrow \quad r = \{\pm i\lambda, \pm\lambda\} \end{aligned}$$

$X(x)$ is simply a linear combination of the e^{rx} terms:

$$X(x) = D_1 e^{-i\lambda x} + D_2 e^{i\lambda x} + D_3 e^{-\lambda x} + D_4 e^{\lambda x}.$$

If we set the constants to be $D_1 = \frac{1}{2}(iC_1 + C_2)$, $D_2 = \frac{1}{2}(-iC_1 + C_2)$, $D_3 = \frac{1}{2}(C_3 - C_4)$, and $D_4 = \frac{1}{2}(C_3 + C_4)$, then we can rewrite $X(x)$ in terms of trigonometric functions. Recall that

$$\begin{aligned} \sin x &= \frac{1}{2i}(e^{ix} - e^{-ix}) \\ \cos x &= \frac{1}{2}(e^{ix} + e^{-ix}) \\ \sinh x &= \frac{1}{2}(e^{x} - e^{-x}) \\ \cosh x &= \frac{1}{2}(e^{x} + e^{-x}). \end{aligned}$$

So we have

$$X(x) = C_1 \sin \lambda x + C_2 \cos \lambda x + C_3 \sinh \lambda x + C_4 \cosh \lambda x.$$

Now we apply the boundary conditions to determine the constants.

$$\begin{aligned} &X(0) = C_2 + C_4 = 0 \\ &X'(0) = \lambda(C_1 + C_3) = 0 \\ &X(\ell) = C_1 \sin \lambda\ell + C_2 \cos \lambda\ell + C_3 \sinh \lambda\ell + C_4 \cosh \lambda\ell = 0 \\ &X'(\ell) = \lambda(C_1 \cos \lambda\ell - C_2 \sin \lambda\ell + C_3 \cosh \lambda\ell + C_4 \sinh \lambda\ell) = 0 \end{aligned}$$

$$C_4 = -C_2 \text{ and } C_3 = -C_1 \rightarrow \begin{cases} C_1(\sin \lambda\ell - \sinh \lambda\ell) + C_2(\cos \lambda\ell - \cosh \lambda\ell) = 0 & (2) \\ C_1(\cos \lambda\ell - \cosh \lambda\ell) - C_2(\sin \lambda\ell + \sinh \lambda\ell) = 0 & (3) \end{cases}$$

Solving (3) for C_1 gives

$$C_1 = \frac{\sin \lambda\ell + \sinh \lambda\ell}{\cos \lambda\ell - \cosh \lambda\ell} C_2$$

and substituting this result into (2) yields

$$C_2 \left[\frac{\sin \lambda\ell + \sinh \lambda\ell}{\cos \lambda\ell - \cosh \lambda\ell}(\sin \lambda\ell - \sinh \lambda\ell) + (\cos \lambda\ell - \cosh \lambda\ell) \right] = 0.$$

$$(\sin \lambda\ell + \sinh \lambda\ell)(\sin \lambda\ell - \sinh \lambda\ell) + (\cos \lambda\ell - \cosh \lambda\ell)^2 = 0$$

$$\sin^2 \lambda\ell - \sinh^2 \lambda\ell + \cos^2 \lambda\ell + \cosh^2 \lambda\ell - 2 \cos \lambda\ell \cosh \lambda\ell = 0$$

$$1 + 1 - 2 \cos \lambda\ell \cosh \lambda\ell = 0$$

$$\cos \lambda\ell \cosh \lambda\ell = 1$$

All positive eigenvalues must satisfy this equation. Let's check to see whether zero is an eigenvalue. The equations to solve are the following.

$$\begin{aligned} T''(t) &= 0 \\ X''''(x) &= 0 \end{aligned}$$

These can be solved by straightforward integration.

$$\begin{aligned} T(t) &= At + B \\ X(x) &= Cx^3 + Dx^2 + Ex + F \end{aligned}$$

With the four boundary conditions, $X(0) = X(\ell) = X'(0) = X'(\ell) = 0$, $X(x) = 0$, so the trivial solution is obtained. Hence, $k = 0$ is not an eigenvalue. As mentioned before, the eigenvalues can't be negative because the solution to $T(t)$ would blow up to infinity as $t \rightarrow \infty$. Therefore, all eigenvalues to the eigenvalue problem satisfy

$$\cos \lambda\ell \cosh \lambda\ell = 1.$$